Comparing Bio-Inspired Algorithms in Constrained Optimization Problems

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***Abstract*— This paper presents a comparison of four bio-inspired algorithms (all seen as search engines) with a similar constraint-handling mechanism (Deb's feasibility rules) to solve constrained optimization problems. The aim is to analyze the performance of traditional versions of each algorithm based on both, final results and on-line behavior. A set of 24 well-known benchmark problems are used in the experiments. Quality and consistency of results per each algorithm are investigated. Furthermore, two performance measures (number of evaluations to reach a feasible solution and progress ratio inside the feasible region) are utilized to compare the on-line behavior of each approach. Based on the obtained results, some conclusions are established.**

I. INTRODUCTION

Evolutionary algorithms (EAs) and other bio-inspired algorithms have been widely used to solve complex search problems, including optimization problems. In the remaining of the paper we will use the name EAs to cover EAs and also other bio-inspired heuristics. Several approaches have been proposed to incorporate feasibility information into the fitness function of an EA in order to solve constrained optimization problems [1]. Most of the research has been focused on designing the adequate constraint-handling [2] and just assuming a search engine (e.g. an EA). On the other hand, in the last decade, novel algorithms, like Differential Evolution (DE) [3] and Particle Swarm Optimization (PSO) [4] have been proposed to solve numerical optimization problems. This paper aims to analyze the behavior of different algorithms but using the same constraint-handling technique (Deb's feasibility rules) in order to study each particular behavior and also to establish some important links, based on a competitive performance, between an specific algorithm and the features of the problem being solved. This experiment is the starting point of some studies related on analyzing EAs more in-depth when dealing with constrained search spaces.

The paper is organized as follows: In Section II we summarize some approaches previously proposed which motivated the present study. Section III presents our proposed analysis. After that, Section IV details the experimental design and presents the results which are also discussed. Finally, in Section V some conclusions are presented and the future paths of research are established.

II. RELATED WORK

Nowadays, besides penalty functions [2], one of the most used constraint-handling mechanisms is the set of feasibility rules proposed by Deb [5]. The rules are the following:

1) Between 2 feasible solutions, the one with the highest fitness value wins.
2) If one solution is feasible and the other one is infeasible, the feasible solution wins.
3) If both solutions are infeasible, the one with the lowest sum of constraint violation is preferred ($\sum_{i=1}^{m} max(0, g_i(\vec{x}))$).

As it may be noted, this approach always prefers feasible solutions over infeasible solutions (i.e. it does not promote diversity), because infeasible solutions with a promising value of the objective function are discarded. However, this technique, coupled with other mechanisms has been widely used in other proposals and the results obtained have been very competitive. We enumerate the following: Mezura et al. [6] used the aforementioned set of rules, coupled with a diversity mechanism using a modified DE/rand/1/bin where each parent vector is able to generate more than one child vector. Takahama and Sakai [7] proposed a ϵ constrained method, which has similarities with Deb's rules. They used DE/rand/1/bin as a search engine, coupled with a gradient-based mutation. Huang et al. [8] proposed a Self-adaptive Differential Evolution for constrained optimization where different DE variants are used, depending of the behavior of the approach. The feasibility rules were used in the replacement procedure. Liang & Suganthan [9] proposed a PSO-based approach, where a set of feasibility rules is used when two particle positions are compared. A dynamic multi-swarm approach is proposed in such a way that each sub-swarm has to either optimize the objective function or satisfy one constraint of the problem. The sub-swarms are assigned adaptively to the constraints, depending of their difficulty. Kukkonen and Lampinen [10] proposed the Generalized DE to solve global and multiobjective optimization problems. The replacement mechanism is modified with a set of rules, similar to Deb's, but emphasizing dominance criterion to select among infeasible solutions and multiple objectives. Brest et al. [11] proposed a self-adaptive DE, where the replacement technique incorporates the feasibility rules. The DE parameters (F and CR) are modified during the run by using random values. Besides, the DE variant is also chosen depending of the behavior in a single run. Zielinski and Laur [12] added the set of feasibility rules in the selection mechanism, to DE, besides a novel proposal to deal with boundary constraints to solve constrained optimization problems. Sinha et al. [13] proposed an approach based on parent centric recombination operator coupled with the feasibility criteria

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and a $(\mu+\lambda)$ replacement to solve constrained problems. On the other hand, other authors have proposed the combination of different search engines, like combining DE and PSO in an approach called Particle Evolutionary Swarm Optimization by Muñoz-Zavala et al. [14] where the DE Mutation is added to a PSO-based approach.

III. Our proposed Analysis

As it may be noted from the methods discussed in the previous Section, the main focus on developing new constraint-handling mechanisms seems to be on adapting a given EA by adding an adequate constraint-handling mechanism. On the other hand, we consider important to study the behavior of some EAs in their more popular variants, with an also popular, simple, effective and parameterless constraint handling mechanism, in order to empirically determine the quality and consistency of their final results. Furthermore, we aim to investigate the behavior of each algorithm in its process to solve a constrained optimization problem. From this analysis, some connections between a search engine and the features of the problem to solve can be established. We believe this information will be useful for practitioners looking for an EA-based technique for constrained optimization. The experiments showed in this paper are part of the initial phase in our study to analyze the behavior of different search engines. We decided to use four algorithms. A real-coded genetic algorithm (GA), a $(\mu+\lambda)$-ES, a global-best particle swarm optimization and the DE/rand/1/bin variant. These were chosen because they are some of the most used in recent works [5], [15], [16], [10]. The pseudocode of each one of the algorithms implemented are presented in Figure 1 for the Differential Evolution, in Figure 2 for the Genetic Algorithm, in Figure 3 for the $(\mu+\lambda)$-ES and finally in Figure 4 for the global-best PSO.

```
Begin
    Generate a random initial population with size= popsize
    Evaluate each vector in the population
    For i=1 to MaxGenerations
        While(j ≤ popsize)
            Parent vector "j" generates one child vector "j'"
            by using DE/rand/1/bin variant
            Evaluate the child vector "j'"
            Compare parent "j"and child "j'" vectors
            using the feasibility rules
            The best will remain for the next generation
             j = j + 1
         end While
    end For
End
```

Fig. 1. DE pseudocode

We know in advance that this kind of comparison has several parameters that may be considered (different encoding techniques, different operators, replacement and selection techniques, plus their corresponding parameters). In this work, we focused our efforts in considering only the most popular EA versions found in the recent specialized literature. We expect this study will give us elements to be addressed in future studies.

```
Begin
    Generate a random initial population with size = popsize
    Evaluate each individual in the population
    For i=1 to MaxGenerations
        While(j ≤ popsize)
            Select 2 individuals from popsize
            using binary tournament based on feasibility rules
            Apply simple arithmetic crossover
            to create two offspring "j" and "j + 1"
            Apply uniform mutation to offspring "j" and "j + 1"
            Evaluate offspring "j" and "j + 1"
             j = j + 2
         end While
        Perform generational replacement
        (offspring remain and all parents are eliminated)
    end For
End
```

Fig. 2. GA pseudocode

```
Begin
    Generate a random initial population with size= μ
    Evaluate each individual in the population
    For i=1 to MaxGenerations
        While(j ≤ λ)
            Select 3 individuals randomly from μ
            Apply simple panmictic recombination
            to create one offspring "j"
            Apply noncorrelated Gaussian mutation
            to offspring "j"
            Evaluate the offspring "j"
             j = j + 1
         end While
        Sort the (μ + λ) individuals
        by using the feasibility rules
        The best μ will remain as
        parents for the next generation
    end For
End
```

Fig. 3. ES pseudocode

```
Begin
    Generate a random initial swarm with size= parsize
    Evaluate each individual in the swarm
    For i=1 to MaxGenerations
        Select the leader of the swarm by using
        the feasibility rules
        While(j ≤ parsize)
            Particle "j" flies by using the
            flight formula with inertia weight
            Evaluate the new position of particle "j"
            Update the pbest value of particle "j"
            using the feasibility rules
             j = j + 1
        end While
    end For
End
```

Fig. 4. PSO pseudocode

IV. EXPERIMENTS AND RESULTS

24 test problems taken from the specialized literature [1], [15], [17] were used to perform the experiments. Because of space restrictions, the full description of the functions is not included, but they can be found in [18] available at http://www.lania.mx/˜emezura/publications/publications-2006.html. A summary of features for each test problem is presented in Table I, where we also show the value of ρ, a metric to estimate the ratio between the feasible region and the entire search space. It was computed as follows: $\rho = |F|/|S|$, where $|F|$ is the number of feasible solutions and $|S|$ is the total number of solutions randomly generated. In this work, $S = 1,000,000$ random solutions.

We defined two experiments: (1) To evaluate the final results obtained by each algorithm based on quality (best result found) and consistency (best mean and standard deviation values) and (2) to analyze the on-line behavior by using two performance measures found in the specialized literature [17]. We performed 30 independent runs for each test function. Equality constraints were transformed into inequalities using a tolerance value of 0.0001.

The parameters used were defined for the four algorithms in such a way that they performed the best and always performing the same number of evaluations (300,000). They are detailed as follows: Real-coded GA: population size $= 100$, generations $= 3000$, crossover rate $= 0.3$, mutation rate $= 0.3$, simple arithmetic crossover, uniform mutation (generating a random value with uniform distribution within the valid interval for a variable), binary tournament selection, generational replacement. $(100 + 300)$-ES: generations $= 750$, panmictic intermediate recombination applied to strategy parameters and decision variables as well, noncorrelated Gaussian mutation. DE/rand/1/bin: population size $= 100$, generations $= 3000$, $CR =$ randomly generated between $(0.8, 1.0)$, $F =$ randomly generated between $(0.3, 0.9)$. Global best PSO: swarm size $= 40$, generations $= 7500$, typical inertia weight flight formula, $C1 = 3.9$, $C2 = 0.1$, $W = 1.5$. These values for the PSO were chosen to increase the influence of the information of each particle ($C1$ and W) and to decrease the influence of the leader ($C2$) as to avoid premature convergence.

The summary of results is presented in Tables II and III. For most of the problems feasible solutions were found either in all 30 independent runs or in none of them. The only exception was GA in problems g14 and g18, where in only 2 and 28 runs, feasible solutions were found respectively.

With the aim to have a more detailed analysis of results, we have grouped the test problems in seven categories. The first includes all 24 test problems, but the remaining six groups have an specific feature: Linear objective function "LOF", nonlinear objective function "NLOF", nonlinear equality constraints "NEC", moderated dimensionality ($n > 5$) "MD", active constraints ($a > 6$) "AC" and small feasible region ($\rho \approx 0$) "SFR". This distribution of problems is presented in Table IV.

The number of problems where the best known result was reached, where the best consistency (best mean and standard deviation values) was obtained and also the number of problems where feasible solutions were found are summarized, per approach, in Table V. For all 24 test problems, DE clearly provided the most competitive results, based on quality and consistency in more problems (19 and 18 respectively) with respect to the rest of the algorithms. DE and ES were able to find feasible solutions in 21 problems. Both, GA and PSO could reach the best known solution in just a few problems (4 and 2 for the GA and 1 and 1 for PSO based on quality and consistency respectively). However, both of them (GA and PSO) were able to find feasible solutions in 16 and 14 problems respectively. For the 6 problems with linear objective function (LOF), there is a very similar performance provided by DE and ES (they had success only in half of the six problems). In fact, ES showed a better consistency in one more problem with respect to DE. GA and PSO provided very poor results again. Regarding problems

TABLE I

MAIN FEATURES OF THE 24 TEST PROBLEMS USED IN THE EXPERIMENTS. n IS THE DIMENSIONALITY OF THE PROBLEM, ρ IS THE ESTIMATED RATIO OF THE FEASIBLE REGION WITH RESPECT TO THE WHOLE SEARCH SPACE, THE NUMBER OF CONSTRAINTS IS DETAILED AS FOLLOWS: LI LINER INEQUALITIES, NI NONLINEAR INEQUALITIES, LE LINEAR EQUALITIES AND NE NONLINEAR EQUALITIES, a IS THE NUMBER OF ACTIVE CONSTRAINTS AT THE BEST KNOWN SOLUTION

Problem	n	Function	ρ	LI	NI	LE	NE	a
$g01$	13	quadratic	0.0111%	9	0	0	0	6
$g02$	20	nonlinear	99.9971%	0	2	0	0	1
$g03$	10	polynomial	0.0000%	0	0	0	1	1
$g04$	5	quadratic	52.1230%	0	6	0	0	2
$g05$	4	cubic	0.0000%	2	0	0	3	3
$g06$	2	cubic	0.0066%	0	2	0	0	2
$g07$	10	quadratic	0.0003%	3	5	0	0	6
$g08$	2	nonlinear	0.8560%	0	2	0	0	0
$g09$	7	polynomial	0.5121%	0	4	0	0	2
$g10$	8	linear	0.0010%	3	3	0	0	6
$g11$	2	quadratic	0.0000%	0	0	0	1	1
$g12$	3	quadratic	4.7713%	0	1	0	0	0
$g13$	5	nonlinear	0.0000%	0	0	0	3	3
$g14$	10	nonlinear	0.0000%	0	0	3	0	3
$g15$	3	quadratic	0.0000%	0	0	1	1	2
$g16$	5	nonlinear	0.0204%	4	34	0	0	4
$g17$	6	nonlinear	0.0000%	0	0	0	4	4
$g18$	9	quadratic	0.0000%	0	13	0	0	6
$g19$	15	nonlinear	33.4761%	0	5	0	0	0
$g20$	24	linear	0.0000%	0	6	2	12	16
$g21$	7	linear	0.0000%	0	1	0	5	6
$g22$	22	linear	0.0000%	0	1	8	11	19
$g23$	9	linear	0.0000%	0	2	3	1	6
$g24$	2	linear	79.6556%	0	2	0	0	2

TABLE IV

THE PROPOSED CLASSIFICATION OF PROBLEMS.

Problems with linear objective function **(LOF)**
$g10$, $g20$, $g21$, $g22$, $g23$, $g24$
Problems with nonlinear objective function **(NLOF)**
$g01$, $g02$, $g03$, $g04$, $g05$, $g06$, $g07$, $g08$, $g09$, $g11$, $g12$, $g13$, $g14$, $g15$, $g16$, $g17$, $g18$, $g19$
Problems with nonlinear equality constraints **(NEC)**
$g03$, $g05$, $g11$, $g13$, $g15$, $g17$, $g20$, $g21$, $g22$, $g23$
Problems with moderated dimensionality ($n \geq 5$) **(MD)**
$g01$, $g02$, $g03$, $g07$, $g09$, $g10$, $g14$, $g17$, $g18$, $g19$, $g20$, 21, $g22$, $g23$
Problems with a moderated number of active constraints ($a \geq 6$) **(AC)**
$g01$, $g07$, $g10$, $g18$, $g20$, $g21$, $g22$, $g23$
Problems with an approximated small feasible region ($\rho \approx 0$) **(SFR)**
$g03$, $g05$, $g11$, $g13$, $g14$, $g15$, $g17$, $g18$, $g20$, $g21$, $g22$, $g23$

TABLE II

STATISTICAL RESULTS OBTAINED ON THE FIRST 12 TEST PROBLEMS IN 30 INDEPENDENT RUNS. A RESULT IN BOLDFACE MEANS EITHER A BETTER RESULT OR BEST KNOW SOLUTION REACHED. "-" MEANS NO FEASIBLE SOLUTIONS WERE FOUND.

Problem & best known solution		**Four algorithms compared**			
		DE	**GA**	**ES**	**PSO**
	Best	**−15.000**	−14.064	−14.972	−14.935
g01	Mean	**−15.000**	−13.982	−14.932	−14.851
−15.000	Std. Dev.	**0**	4.6E-2	3.0E-2	4.3E-2
	Best	**−0.803619**	−0.768219	−0.803535	−0.591656
g02	Mean	**−0.797815**	−0.744961	−0.778222	−0.511174
−0.803619	Std. Dev.	**9.0E-3**	1.7E-2	2.2E-2	2.6E-2
	Best	−0.820	−0.348	**−0.832**	−0.808
g03	Mean	**−0.487**	−0.075	−0.393	−0.165
−1.000	Std. Dev.	1.3E-1	**9.9E-2**	1.3E-1	2.4E-1
	Best	**−30665.539**	−30654.531	**−30665.539**	−30637.414
g04	Mean	**−30665.539**	−30582.522	**−30665.539**	−30613.192
−30665.539	Std. Dev.	**0**	4.0E+1	**0**	1.1E+1
	Best	**5126.497**	–	5128.490	–
g05	Mean	**5127.531**	–	5188.059	–
5126.497	Std. Dev.	**5.4E+0**	–	4.7E+1	–
	Best	**−6961.814**	−6846.993	**−6961.814**	−6959.643
g06	Mean	**−6961.814**	−6303.951	**−6961.814**	−6932.836
−6961.814	Std. Dev.	**0**	285.186118	**0**	1.7E+1
	Best	**24.306**	25.228	**24.306**	92.002
g07	Mean	**24.306**	44.456	**24.306**	172.694
24.306	Std. Dev.	**1.0E-7**	2.5E+1	1.0E-6	3.8E+1
	Best	**−0.095825**	**−0.095825**	**−0.095825**	−0.095823
g08	Mean	**−0.095825**	**−0.095825**	**−0.095825**	−0.095727
−0.095825	Std. Dev.	**0**	1.0E-6	**0**	8.0E-5
	Best	**680.630**	682.284	**680.630**	697.763
g09	Mean	**680.630**	689.766	**680.630**	730.006
680.630	Std. Dev.	**0**	6.8	**0**	1.7E+1
	Best	**7049.248**	7226.913	**7049.248**	9700.121
g10	Mean	**7049.248**	9469.818	**7049.248**	10524.581
7049.248	Std. Dev.	**0**	1.5E+3	3.0E-5	4.3E+2
	Best	**0.750**	**0.750**	**0.750**	**0.750**
g11	Mean	**0.750**	0.819	0.792	0.751
0.750	Std. Dev.	**0**	8.7E-2	5.4E-2	1.1E-3
	Best	**−1.000**	**−1.000**	**−1.000**	**−1.000**
g12	Mean	**−1.000**	**−1.000**	**−1.000**	**−1.000**
−1.000	Std. Dev.	**0**	**0**	**0**	2.4E-5

TABLE III

STATISTICAL RESULTS OBTAINED ON THE LAST 12 TEST PROBLEMS IN 30 INDEPENDENT RUNS. A RESULT IN BOLDFACE MEANS EITHER A BETTER RESULT OR BEST KNOW SOLUTION REACHED. "-" MEANS NO FEASIBLE SOLUTIONS WERE FOUND. (n) MEANS THE NUMBER OF RUNS (OUT OF 30) WHERE FEASIBLE SOLUTIONS WERE FOUND.

Problem & Best know solution		**Four algorithms compared**			
		DE	**GA**	**ES**	**PSO**
	Best	**0.147885**	–	0.438878	–
g13	Mean	**0.453048**	–	0.896232	–
0.053942	Std. Dev.	**8.7E-2**	–	1.7E-1	–
	Best	**−47.765**	−45.038 (2)	**−47.765**	–
g14	Mean	**−47.765**	−44.925 (2)	**−47.765**	–
−47.765	Std. Dev.	**0**	1.6E-1 (2)	**0**	–
	Best	**961.715**	–	**961.715**	–
g15	Mean	**961.715**	–	961.949	–
961.715	Std. Dev.	**6.0E-6**	–	4.3E-1	–
	Best	**−1.905**	−1.899	**−1.905**	−1.888
g16	Mean	**−1.905**	−1.782	**−1.905**	−1.867
−1.905	Std. Dev.	**0**	8.7E-2	**0**	9.7E-3
	Best	**8853.540**	–	8859.511	–
g17	Mean	**8920.605**	–	8935.020	–
8853.540	Std. Dev.	3.2E+1	–	**3.1E+1**	–
	Best	**−0.866025**	−0.852457 (28)	**−0.866025**	–
g18	Mean	**−0.866025**	−0.590087 (28)	−0.866024	–
−0.866025	Std. Dev.	**0**	1.3E-1 (28)	1.0E-6	–
	Best	**32.656**	102.080	32.734	67.820
g19	Mean	**32.656**	177.375	32.813	81.964
32.656	Std. Dev.	**2.21E-5**	3.5E+1	4.6E-2	7.8
	Best	–	–	–	–
g20	Mean	–	–	–	–
0.096737	Std. Dev.	–	–	–	–
	Best	**193.725**	–	**193.725**	–
g21	Mean	193.764	–	**193.725**	–
193.725	Std. Dev.	2.1E-1	–	**6.0E-5**	–
	Best	–	–	–	–
g22	Mean	–	–	–	–
236.431	Std. Dev.	–	–	–	–
	Best	–	–	–	–
g23	Mean	–	–	–	–
400.055	Std. Dev.	–	–	–	–
	Best	**−5.508**	**−5.508**	**−5.508**	−5.507
g24	Mean	**−5.508**	−5.494	**−5.508**	−5.501
−5.508	Std. Dev.	**0**	1.4E-2	**0**	4.1E-3

with nonlinear objective function (NLOF), DE and ES found feasible solutions in all 18 problems, but DE showed a better quality and robustness in 17 problems, while the best known solution was reached by ES in 12 problems and competitive mean and St. deviation values were obtained in only 9 problems. GA and PSO found feasible solutions in 14 and 12 problems respectively, but failed to be competitive in most of the 18 problems.

For the 10 problems with nonlinear equality constraints (NEC), together again, DE and ES found feasible solutions in 7 of them, while GA and PSO reached the feasible region in only 2. The best solution was reached by DE in 6 problems, by ES in 4, by GA in only 1 and by PSO in 0. The best consistency was obtained by DE in 4 problems, ES only in 1 and GA and PSO in 0 problems.

DE and ES found feasible solutions in 11 of 14 problems with a moderated high dimensionality (MD) and GA and PSO in 9 and 7 functions respectively. DE obtained the best known solution in 10 problems, ES in 6, and GA and PSO could not find the vicinity of the best solution in none of them. DE provided the best consistency in 8 problems, while ES was better in this issue in 6 test functions. GA and PSO, again, could not reach consistent competitive results in any of these problems.

For the 8 problems with more than 6 active constraints (AC) , DE and ES reached the feasible region in 5 problems, GA in 4 and PSO in 3. DE reached the best known solution in 5 problems, ES in 4 and GA and PSO in 0. The most consistent behavior was provided by DE in 4 problems, by ES in 3 problems, and in 0 by GA and PSO.

Finally, for the 12 problems with a very small feasible region with respect to the whole search space, DE and ES found feasible solutions in 9 of them, GA in 4 and PSO in 2. DE found the best solution in 8 problems, ES in 5 and GA and PSO in just 1 function. The most consistent results were found by DE in 6 problems, by ES in 2 and by GA and PSO in none of the problems.

The overall results provide the following findings:

- All four approaches were able to find feasible solutions in almost all problem categories. The exception was the category of problems with nonlinear equality constraints, where GA and PSO presented some problems to do it.
- In problems with a linear objective function, regardless of the type and number of constraints, ES showed a better consistency with respect to the rest of the algorithms compared.
- DE and ES are able to reach the feasible region with a similar performance.
- DE provided the most competitive results, based on quality and consistency in problems with nonlinear objective function.
- It was difficult to find good quality results and also some consistency for all four approaches in problems with nonlinear equality constraints. This finding agrees with previous studies [17].

TABLE V

SUMMARY OF FINDINGS FROM TABLES II AND III. A RESULT IN BOLDFACE MEANS A BETTER RESULT.

All problems (24 problems)				
DE	GA	ES	PSO	
19	4	14	1	Best result found
18	2	11	1	Best consistency
21	16	**21**	14	Feasible Sol. found
LOF (6 problems)				
3	1	3	0	Best result found
2	0	**3**	0	Best consistency
3	2	**3**	2	Feasible Sol. found
NLOF (18 problems)				
17	3	12	2	Best result found
17	2	9	1	Best consistency
18	14	**18**	12	Feasible Sol. found
NEC (10 problems)				
6	1	4	0	Best result found
4	0	1	0	Best consistency
7	2	**7**	2	Feasible Sol. found
MD (14 problems)				
10	0	6	0	Best result found
8	0	6	0	Best consistency
11	9	**11**	7	Feasible Sol. found
AC (8 problems)				
5	0	4	0	Best result found
4	0	3	0	Best consistency
5	4	**5**	3	Feasible Sol. found
SFR (12 problems)				
8	1	5	1	Best result found
6	0	2	0	Best consistency
9	4	**9**	2	Feasible Sol. found

- A moderated dimensionality affected GA and PSO. ES was less affected and the most robust approach when dealing with multiple variables was DE.
- GA and PSO performances were affected in problems with more than six active constraints.
- DE and ES were able to provide a competitive performance in problems with an estimated small feasible region. In contrast, GA and PSO were clearly affected in their corresponding behaviors.

In our second experiment, we analyzed two aspects in the behavior of each of the four compared approaches. Then, two performance measures found in the specialized literature were used.

The first performance measure was used by Lampinen [19] to count how many evaluations are required for a given approach to find the first feasible solution; it was called EVALS in [17]. The second one was proposed in [17] to measure the capability of a given approach to improve feasible solutions. It was called progress ratio because it measures the progress inside the feasible region. The corresponding expression is the following:

$$\text{Progress ratio} = \left| \ln \sqrt{\frac{f_{\min}(G_{\text{ff}})}{f_{\min}(T)}} \right| \tag{1}$$

where $f_{\min}(G_{\text{ff}})$ refers to the objective function value of the first feasible solution found and $f_{\min}(T)$ refers to the objective function value of the best feasible solution found in the last generation.

TABLE VI

STATISTICAL RESULTS FOR THE EVALS PERFORMANCE MEASURE ON 30 INDEPENDENT RUNS FOR THE FIRST 12 TEST PROBLEMS. "-" MEANS NO FEASIBLE SOLUTIONS FOUND. BETTER RESULT IN BOLDFACE

Prob.		EVALS			
		DE	GA	ES	PSO
g01	Best	1,775	1,013	400	**167**
	Mean	2,411	3,891	1,987	**239.6**
	Std. Dev.	2.7E+2	2.2E+3	1.1E+3	**4.4E+1**
g02	Best	**0**	**0**	**0**	**0**
	Mean	**0**	**0**	**0**	**0**
	Std. Dev.	**0**	**0**	**0**	**0**
g03	Best	623	597	3,200	**393**
	Mean	5,210	**4,176**	15,320	143,655
	Std. Dev.	3.1E+3	**2.8E+3**	1.1E+4	8.E+4
g04	Best	**0**	**0**	**0**	**0**
	Mean	**0**	**0**	31	**0**
	Std. Dev.	**0**	**0**	3.0E+0	**0**
g05	Best	**16,791**	–	62,500	–
	Mean	**20,601**	–	72,633	–
	Std. Dev.	**2.3E+3**	–	4.6E+3	–
g06	Best	**0**	119	100	103
	Mean	1,052	8,739	2,380	**323**
	Std. Dev.	4.6E+2	1.4E+4	9.7E+2	**1.6E+1**
g07	Best	968	**515**	3,600	1,312
	Mean	**2,024**	2,453	6,040	9,441
	Std. Dev.	**5.1E+2**	8.1E+2	1.4E+3	6.4E+3
g08	Best	**0**	**0**	1	**0**
	Mean	60	**55**	160	73
	Std. Dev.	7.6E+1	1.1E+2	1.6E+2	**4.2E+1**
g09	Best	**0**	**0**	8	**0**
	Mean	**38**	174	308	91
	Std. Dev.	5.1E+1	1.6E+2	**1.3E+1**	7.2E+1
g10	Best	1,312	1,260	5,200	**208**
	Mean	**2,803**	16,449	8,133	7,357
	Std. Dev.	**6.1E+2**	1.8E+4	1.5E+3	4.6E+3
g11	Best	122	451	1,600	**87**
	Mean	2034	15,423	9,080	**974**
	Std. Dev.	9.3E+2	2.2E+4	4.7E+3	**8.0E+2**
g12	Best	**0**	**0**	1	**0**
	Mean	1	**0**	23	4
	Std. Dev.	8.0E+0	**0**	2.1E+1	1.0E+1

TABLE VII

STATISTICAL RESULTS FOR THE EVALS PERFORMANCE MEASURE ON 30 INDEPENDENT RUNS FOR THE LAST 12 TEST PROBLEMS. "-" MEANS NO FEASIBLE SOLUTIONS FOUND. BETTER RESULT IN BOLDFACE

Prob.		EVALS			
		DE	GA	ES	PSO
g13	Best	**27,929**	–	100,000	–
	Mean	**42,180**	–	133,107	–
	Std. Dev.	**8.1E+3**	–	1.9E+4	–
g14	Best	**26,969**	170,649	69,600	–
	Mean	**30,424**	209,927	109,493	–
	Std. Dev.	**2.1E+3**	5.5E+4	2.1E+4	–
g15	Best	**9,880**	–	37,600	–
	Mean	**14,174**	–	49,933	–
	Std. Dev.	**2.7E+3**	–	6.7E+3	–
g16	Best	**0**	**0**	500	70
	Mean	960	1,539	2,780	**86**
	Std. Dev.	**4.3E+2**	1.2E+3	4.4E+2	4.4E+2
g17	Best	**32,270**	–	104,100	–
	Mean	**41,869**	–	130,140	–
	Std. Dev.	**4.8E+3**	–	1.1E+4	–
g18	Best	**5,109**	5,120	14,400	–
	Mean	**6,443**	25,702	21,560	–
	Std. Dev.	**7.9E+2**	3.0E+4	2.3E+3	–
g19	Best	**0**	**0**	**0**	**0**
	Mean	**0**	**0**	1.7	**0**
	Std. Dev.	**0**	**0**	1.0E+0	**0**
g20	Best	–	–	–	–
	Mean	–	–	–	–
	Std. Dev.	–	–	–	–
g21	Best	**27,101**	–	71,100	–
	Mean	**30,737**	–	80,687	–
	Std. Dev.	**2.7E+3**	–	4.1E+3	–
g22	Best	–	–	–	–
	Mean	–	–	–	–
	Std. Dev.	–	–	–	–
g23	Best	–	–	–	–
	Mean	–	–	–	–
	Std. Dev.	–	–	–	–
g24	Best	**0**	**0**	**0**	**0**
	Mean	**0**	**0**	2	**0**
	Std. Dev.	**0**	**0**	2.0E+0	**0**

We performed 30 independent runs per approach per test problem to calculate the value for each one of the two performance measures. The parameter values used for each one of the approaches are exactly the same used in the previous experiment. The summary of results is presented, for the EVALS performance measure in TablesVI and VII, and for the Progress ratio in Tables VIII and IX.

Based on our proposed classes of problems, we present the following discussion with respect to the EVALS performance measure: For problems with a linear objective function (LOF) DE was the algorithm with the fastest approach to the feasible region overall. However, PSO was able to do it faster than DE in problems g10 (8 variables, 3 linear inequality and 3 nonlinear inequality constraints). Regarding problems with a nonlinear objective function (NLOF), DE was the most competitive based on EVALS measure. However, GA reached the feasible region with an equal or a lower number of evaluations in problems g03 (10 decision variables and 1 nonlinear equality constraint) and g07 (10 variables, 3 linear inequality and 5 nonlinear inequality constraints). Furthermore, in problems g01 (13 variables and 9 linear equality constraints), g06 (2 variables and 2 nonlinear inequality constraints), g11 (2 variables and 1 nonlinear equality constraint) and g16 (5 variables, 4 linear inequality and 34 nonlinear inequality constraints), PSO provided a more robust approach to the feasible region. For problems with nonlinear equality constraints (NEC), DE provided the most competitive values for EVALS overall. However GA showed more consistency in problem g03 and PSO in problem g11. While DE was, again, more competitive in problems with a moderated dimensionality (MD), GA provided an equal or even better performance in problems g03 and g07. Also, PSO was superior in its best EVALS value in problem g10. For the set of test functions where there are more than six active constraints (AC) PSO reached the feasible region faster than DE in problems g01 and g10. The same behavior was observed by GA over DE in problem g07. In the remaining functions, DE was better. Finally, in problems with a very small feasible region, DE was the most competitive algorithm, but GA was more consistent in problem g03 and PSO in problem g11. It is worth noticing that ES, which provides very competitive final results in the previous experiments, requires more evaluations to reach the feasible region. It may be due to its self-adaptation mechanism for the mutation operator; but this issue requires a more in-depth analysis.

Regarding the Progress Ratio measure, the following findings are presented: For LOF problems PSO was better in problem g10, ES in problem g21 (7 variables, 1 nonlinear

TABLE VIII

STATISTICAL RESULTS FOR THE PROGRESS RATIO PERFORMANCE MEASURE ON 30 INDEPENDENT RUNS FOR THE FIRST 12 TEST PROBLEMS. "-" MEANS NO FEASIBLE SOLUTIONS FOUND. BETTER RESULT IN BOLDFACE

Prob.		Progress Ratio			
		DE	**GA**	**ES**	**PSO**
g01	Best	**1.947**	1.871	1.429	1.273
	Mean	**1.118**	0.772	1.098	0.506
	Std. Dev.	3.5E-1	3.2E-1	**2.6E-1**	3.4E-1
g02	Best	1.365	**1.421**	1.327	1.068
	Mean	1.060	**1.079**	1.038	0.837
	Std. Dev.	1.3E-1	1.5E-1	1.1E-1	**1.0E-1**
g03	Best	3.683	5.804	4.287	**9.0989**
	Mean	1.305	1.934	2.035	**5.799**
	Std. Dev.	**9.3E-1**	1.5E+0	1.3E+0	2.1E+0
g04	Best	0.119	0.110	**0.129**	0.124
	Mean	0.064	**0.072**	0.059	0.055
	Std. Dev.	3.1E-2	**2.3E-2**	3.4E-2	4.0E-2
g05	Best	0.033	–	**0.766**	–
	Mean	0.010	–	**0.175**	–
	Std. Dev.	**8.0E-3**	–	1.7E-1	–
g06	Best	0.836	0.737	3.158	**3.323**
	Mean	0.407	0.261	**1.113**	0.566
	Std. Dev.	**2.0E-1**	2.1E-1	1.0E+0	8.1E-1
g07	Best	2.490	2.392	**2.648**	1.648
	Mean	1.968	1.536	**2.197**	0.941
	Std. Dev.	**2.8E-1**	3.9E-1	3.0E-1	3.3E-1
g08	Best	4.538	3.400	**11.903**	7.267
	Mean	1.044	0.840	**2.207**	1.680
	Std. Dev.	1.2E+0	**1.0E+0**	2.5E+0	1.8E+0
g09	Best	**4.777**	4.698	4.390	4.718
	Mean	2.085	2.515	**2.612**	2.569
	Std. Dev.	1.3E+0	1.5E+0	**1.2E+0**	1.4E+0
g10	Best	0.672	0.635	0.608	**3.323**
	Mean	0.537	0.393	0.389	**0.566**
	Std. Dev.	**8.3E-2**	9.9E-2	1.6E-1	8.1E-1
g11	Best	0.144	0.125	0.913	**1.720**
	Mean	0.089	0.031	**0.413**	0.347
	Std. Dev.	4.6E-2	**3.8E-2**	2.6E-1	3.5E-1
g12	Best	0.311	0.311	**0.507**	**0.507**
	Mean	0.104	0.107	0.141	**0.169**
	Std. Dev.	**5.8E-2**	7.8E-2	7.3E-2	1.1E-1

TABLE IX

STATISTICAL RESULTS FOR THE PROGRESS RATIO PERFORMANCE MEASURE ON 30 INDEPENDENT RUNS FOR THE LAST 12 TEST PROBLEMS. "-" MEANS NO FEASIBLE SOLUTIONS FOUND. BETTER IN BOLDFACE

Prob.		Progress Ratio			
		DE	**GA**	**ES**	**PSO**
g13	Best	0.921	–	**13.191**	–
	Mean	0.390	–	**1.824**	–
	Std. Dev.	**1.4E-1**	–	2.8E+0	–
g14	Best	0.081	0	**1.709**	–
	Mean	0.055	0	**1.515**	–
	Std. Dev.	**1.3E-2**	0	1.4E-1	–
g15	Best	0.002	–	**0.261**	–
	Mean	0.001	–	**0.068**	–
	Std. Dev.	**1.0E-3**	–	5.6E-2	–
g16	Best	0.359	0.363	2.844	**3.266**
	Mean	0.230	0.203	**0.938**	0.673
	Std. Dev.	**5.5E-2**	7.2E-2	6.8E-1	7.5E-1
g17	Best	0.002	–	**0.040**	–
	Mean	0.001	–	**0.017**	–
	Std. Dev.	**2.7E-4**	–	8.6E-3	–
g18	Best	1.354	**2.081**	0.312	–
	Mean	**0.774**	0.687	0.312	–
	Std. Dev.	2.6E-1	4.2E-1	**0**	–
g19	Best	**3.587**	2.755	3.511	3.115
	Mean	**3.142**	2.257	3.074	2.707
	Std. Dev.	3.0E-1	2.7E-1	3.4E-1	**1.9E-1**
g20	Best	–	–	–	–
	Mean	–	–	–	–
	Std. Dev.	–	–	–	–
g21	Best	0.814	–	**1.726**	–
	Mean	**0.631**	–	0.573	–
	Std. Dev.	**1.6E-1**	–	3.6E-1	–
g22	Best	–	–	–	–
	Mean	–	–	–	–
	Std. Dev.	–	–	–	–
g23	Best	–	–	–	–
	Mean	–	–	–	–
	Std. Dev.	–	–	–	–
g24	Best	1.094	**1.902**	1.224	1.535
	Mean	0.013	**0.546**	0.261	0.261
	Std. Dev.	**2.2E-1**	4.3E-1	2.3E-1	3.3E-1

inequality and 5 nonlinear equality constraints) and GA in problem g24 (2 variables and 2 nonlinear inequality constraints). For NLOF problems, ES was the most competitive (problems g04, g05, g07, g08, g12, g13, g14, g15 and g17), followed by PSO (problems g03, g06, g11, g12 and g16). For NEC problems, ES was the most competitive in most of the problems (g05, g13, g15 and g17). For MD problems, ES provided better results in most of the problems (g07, g14, g17 and g21), followed by DE in problems g01, g09 and g19, PSO in problems g03 and g10 and GA in problem g02. In AC problems, DE was the most competitive in problems g01 and g21, ES in problem g07 and PSO in problem g10. Finally, for SFR problems, ES provided the best values in problems g05, g13, g14, g15, g17 and g21, PSO in problems g03 and g11 and GA in problem g18.

From the overall results obtained in the second experiment, we observed the following:

- The feasible region is reached by DE faster than ES, GA and PSO. However, in problems with 10 variables with a small feasible region, GA was superior. Furthermore, PSO found the first feasible solution faster than DE in problems with different features (from the results it is difficult to determine the type of problems where PSO performed better than DE).
- The results suggest that ES requires more time than the other three algorithms to reach the feasible region. In fact, PSO, with its well-known fast convergence, is able to be also faster than other algorithms to find a feasible solution for some constrained problems.
- For problems with nonlinear objective function ES was the most capable algorithm to improve feasible solutions. PSO was the second best.
- In problems with nonlinear equality constraints it was difficult for all approaches to improve feasible solutions (though ES was the best). This lead us to suggest that, most of the time, the first feasible solution found by an algorithm, in the best scenario, is just slightly improved. Some local search mechanisms could be an interesting solution to this issue.
- A moderated dimensionality slightly affected the capabilities of DE and ES to improve feasible solutions. GA and PSO were more affected but they did not lose this ability.
- In problems with a small feasible region, ES was the most competitive on improving feasible solutions.

From the two experiments, we present the following remarks: (1) The four algorithms are able to find feasible solutions in most of the problems. (2) DE seems to be the most suitable algorithm for the constraint-handling mechanism based on the feasibility rules, followed by ES. However, they work in different ways i.e. DE reaches the feasible region faster with a low improvement inside the feasible region. In contrast, ES requires more time to generate a feasible solution, but its improvement inside the feasible region is higher. (3) The main source of difficulty is the number of nonlinear equality constraints for all algorithms. (4) PSO is able to find feasible solutions faster than GA and ES and, in some problems, also than DE. (5) ES showed a very competitive performance to improve feasible solutions previously chosen, followed by PSO. (6) It is interesting to note that both, DE and ES (the most competitive algorithms) apply the feasibility rules in the replacement phase and use a panmictic operator. These findings are far from being conclusive. However, they provide a starting point from further research, which is detailed in the next Section.

V. Conclusions and Future Work

An empirical comparison of four popular versions of EAs, coupled with one of the most used constraint-handling mechanism nowadays has been presented. DE/rand/1/bin, a real-coded GA, a $(\mu + \lambda)$-ES and a global-best PSO were implemented. The constraint-handling mechanism added to them was the set of feasibility rules proposed by Deb [5]. The experimental design allowed to analyze several aspects of each approach: quality and consistency of final results, the number of evaluations required to reach the feasible region and the capability to improve the first feasible solution (how easy is for a given approach to move inside the feasible region). The overall results suggest that all compared approaches were able to find the feasible region for most of the test problems, but DE was the most competitive approach in this set of test functions. Besides, PSO showed some competitive results when reaching the feasible region faster than the other three approaches and ES presented a higher performance to improve feasible solutions previously found, followed by PSO and DE. Finally, the main source of difficulty for all four algorithms was a high number of nonlinear equality constraints. The aforementioned conclusions from this initial experiment lead us to the following paths of research: (1) To analyze each algorithm by using different operators (previously used in other approaches for constrained optimization), (2) to investigate more in-depth different DE variants as to determine its main elements for its success and (3) to study the interesting capability of the ES to move inside the feasible region, because it seems to be one of the features most difficult to find in an EA for constrained optimization.

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